

This paper studies the time complexity of deciding graph properties definable by sentences of first-order logic in prenex normal form with k quantified variables. The trivial algorithm for this problem runs in time $O(n^k)$ on graphs of n vertices.

The paper presents the first algorithms for the general case, i.e., for arbitrary graphs and arbitrary first-order sentences, that is faster than the trivial algorithm. One algorithm runs in time $O(n^{k-3+\omega})$ for all $k \geq 3$, where $\omega < 2.373$ is the matrix multiplication exponent, i.e., the smallest number ω such that multiplication of two $n \times n$ -matrices can be computed in time $O(n^\omega)$. Another algorithm based on fast rectangular matrix multiplication runs in time $n^{k-1+o(1)}$ for all $k \geq 9$.

Finally it is observed that assuming the strong exponential time hypothesis, i.e., the assumption that SAT cannot be solved in time $(2 - \epsilon)^n$ for some $\epsilon > 0$, there is no algorithm that solves the problem generally in time less than $O(n^{k-1})$.