The paper defines a hierarchy of bounded arithmetic theories that form a stratification of the conservative extension S_1 of $I\Delta_0$. The theory S_1 is formulated in the language of Buss' bounded arithmetic S_2, but without the function $\#$ which makes the growth rate of terms polynomial instead of just linear. Such a stratification should be such that the theories coincide with the well-known hierarchy of Buss' theories of Bounded Arithmetic in the presence of terms of polynomial growth rate, and naturally correspond to meaningful computational complexity classes.

The theories introduced are $\breve{S}^i_k$ and $\breve{T}^i_k$ and TLS^i_k and TSC^i_k for $i \geq 0$ and $k = 1, 2$. The theories $\breve{S}^i_2$ and $\breve{T}^i_2$ are equivalent to Buss' theories S^i_2 and T^i_2, respectively.

For the axiomatization of the theories, certain subclasses $\tilde{\Sigma}^b_i$ and $\breve{\Sigma}^b_i$ of Σ^b_i-formulas, and the corresponding subclasses of Π^b_i are defined. (I am simplifying a bit over the notation used in the paper.) The theories $\breve{T}^i_k$ and $\breve{S}^i_k$ are defined by a set of basic axioms plus induction, resp. length induction, for $\breve{\Sigma}^b_i$-formulas. The theories TLS^i_k and TSC^i_k are axiomatized by open induction plus an iteration scheme for certain subclasses of $\breve{\Sigma}^b_i$-formulas.

As usual, we say that a predicate is $\tilde{\Delta}_i$ in a theory T if T proves it is equivalent to both a $\tilde{\Sigma}^b_i$-formula and a $\tilde{\Pi}^b_i$ formula. A multifunction (search problem) is $\tilde{\Sigma}_i$-definable in a theory T if its graph is defined by a $\tilde{\Sigma}^b_i$-formula, and T proves its totality.

It is shown that these theories form a hierarchy $TLS^i_k \subseteq TSC^i_k \subseteq \breve{S}^i_k \subseteq TLS^{i+1}_k$, and thus their union is S_k. Moreover it is shown that:

- The predicates that are $\tilde{\Delta}^b_1$ in TLS^1_1 are precisely those in Logspace.
- The predicates that are $\tilde{\Delta}^b_1$ in TSC^1_1 are precisely those in the complexity class SC.

For the theories higher up in the theories, relativized versions of these characterizations hold.

- The predicates that are $\tilde{\Delta}^b_{i+1}$ in TLS^{i+1}_1 are precisely those in Logspace with an oracle for a predicate in $\tilde{\Sigma}^b_i$.
- The predicates that are $\tilde{\Delta}^b_{i+1}$ in TSC^{i+1}_1 are precisely those in the complexity class SC with an oracle for a predicate in Σ.

Similar relations hold for the provably total multifunctions, i.e. total search problems, of these theories.

- The multifunctions that are $\tilde{\Sigma}^b_1$-definable in TLS^1_1 are precisely those in Logspace.
- The multifunctions that are $\tilde{\Sigma}^b_1$-definable in TSC^1_1 are precisely those in the complexity class SC.

Also, for the theories higher up in the hierarchy, relativized versions of these results hold, making use of witness oracles.

- The multifunctions that are $\tilde{\Sigma}^b_{i+1}$-definable in TLS^{i+1}_1 are precisely those in Logspace with a witness-oracle for a predicate in $\tilde{\Sigma}^b_i$.
- The multifunctions that are $\tilde{\Sigma}^b_{i+1}$-definable in TSC^{i+1}_1 are precisely those in the complexity class SC with a witness-oracle for a predicate in $\tilde{\Sigma}^b_i$.

It is also shown that the theory TLS^{i+1}_k is conservative over $\breve{S}^i_k$ w.r.t. boolean combinations of $\tilde{\Sigma}^b_{i+1}$-formulas.

I was not able to verify all the proofs in the paper, which are mostly in the appendix, but they are very plausibly correct. The definitions of the formula classes and theories used in the paper are very complex and involved, and designed to obviously be able to capture the kinds of computations they are meant to characterize. Therefore while the goal of providing a stratification of the theory S_1 with the required properties is achieved, the fragments defined can hardly be considered natural. Therefore the results in the paper are of rather limited interest, and probably only for experts in the field of bounded arithmetic.

The notation in the paper is very cumbersome, and the author would be well advised to simplify and streamline it.